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Lagrangian Formalism for Multiform Fields on Minkowski Spacetime

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We present the Lagrangian formalism for multiform fields on Minkowski spacetime based on the multiform and extensor calculus. The formulation gives a unified mathematical description for the relativistic field theories including the gravitational field. We work out examples including the Dirac–Hestenes field on the gravitational background.

1. INTRODUCTION

The multiform and *extensor* fields over Minkowski spacetime provide a unifying language for the field equations, including gravitation (Hestenes, 1966; Hestenes and Sobczyk, 1984; Rodrigues and de Souza, 1993, 1994; Moya 1999; Moya *et al.*, 2000a). A review, including new mathematical topics, is in preparation (Moya *et al.*, 1999a, b). A multiform Lagrangian formalism using rigorous mathematics is lacking, despite some previous attempts (Lasenby *et al.*, 1993; Rodrigues and de Souza, 1994; Rodrigues *et al.*, 1995). In this paper, we provide such a theory. In our formalism, different kinds of Lagrangians which occurs in physical theories are treated with the same mathematics. We include the identities (the tricks of the trade) necessary for the derivation of equations of motion. In Moya *et al.*, (2000a, b), we present a Lagrangian formulation for the gravitational field as a distortion field (an extensor field) on Minkowski spacetime. There, we show that the

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formulation of the same problem by Lasenby *et al.*, (1998) is a non sequitur. The gravitational field as a distortion field appears in Rodrigues and de Souza (1993).

In the sequel, M is a 4-dimensional manifold diffeomorphic to R^4 oriented by the volume element 4-form τ_η and time-oriented. A Lorentzian flat metric is $\boldsymbol{\eta} \in \sec T_2^0(M)$ and $D^{\boldsymbol{\eta}}$ is the Levi-Civita connection of $\boldsymbol{\eta}$ (Sachs and Wu, 1977). The multiform calculus on Minkowski spacetime $(M, \boldsymbol{\eta}, \tau_{\boldsymbol{\eta}}, D^{\boldsymbol{\eta}})$ is enhanced once we use an affine space $(M, \mathcal{M}^*)$, where $\mathcal{M}^*$, the dual of $\mathcal{M} \approx R^4$, is the vector space of the structure.

Given a global coordinate system over $M \ni x \leftrightarrow x^\mu(x) \in R$, $\mu = 0, 1, 2, 3$, associated to a inertial reference frame (Rodrigues and Rosa, 1989) at $x \in M$, $\langle \partial/\partial x^\mu|_x\rangle$ and $\langle dx^\mu|_x\rangle$ are the bases for the tangent vector space T_xM and the tangent covector space T_x^*M,

$$\boldsymbol{\eta} = \eta_{\mu\nu}\, d_x^\mu \otimes dx^\nu, \qquad \eta_{\mu\nu} = \boldsymbol{\eta}\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right) = diag(1, -1, -1, -1)$$

Definition 1.1 (Minkowski space). $T_xM \ni \mathbf{v}_x$ is said to be equivalent to $\mathbf{v}_{x'} \in T_{x'}M$, written $\mathbf{v}_x = \mathbf{v}_{x'}$, if and only if

$$\boldsymbol{\eta}_{(x)}\left(\left.\frac{\partial}{\partial x^\mu}\right|_x, \mathbf{v}_x\right) = \boldsymbol{\eta}_{(x')}\left(\left.\frac{\partial}{\partial x^\mu}\right|_{x'}, \mathbf{v}_{x'}\right), \qquad \left.\frac{\partial}{\partial x^\mu}\right|_x = \left.\frac{\partial}{\partial x^\mu}\right|_{x'}$$

The set of equivalent classes of tangent vectors over the tangent bundle, $\mathcal{M} = \{\mathscr{C}_{\mathbf{v}_x}|x \in M\}$, is a vector space, called Minkowski vector space.

A natural basis for $\mathcal{M}$ is $\langle \mathscr{C}_{\partial/\partial x^\mu|_x}\rangle$. With $\vec{v} \equiv \mathscr{C}_{\mathbf{v}_x}$ and $\vec{e}_\mu \equiv \mathscr{C}_{\partial/\partial x^\mu|_x}$, we can write $\vec{v} = v^\mu\, \vec{e}_\mu$.

Definition 1.2. The dual basis of $\langle \vec{e}_\mu\rangle$ is $\langle\gamma^\mu\rangle$, i.e., $\gamma^\mu \in \mathcal{M}^* \equiv \Lambda^1(\mathcal{M})$ and $\gamma^\mu(\vec{e}_\nu) = \delta^\mu_\nu$.

Definition 1.3. The 2-tensor over $\mathcal{M}$, η: $\mathcal{M} \times \mathcal{M} \to R$, such that for each $\vec{v} = \mathscr{C}_{\mathbf{v}_x}$ and $\vec{w} = \mathscr{C}_{\mathbf{w}_x} \in \mathcal{M}$: $\eta(\vec{v}, \vec{w}) = \eta_{(x)}(\mathbf{v}_x, \mathbf{w}_x)$, for all $x \in M$, is called a Minkowski metric tensor.

For $\vec{e}_\mu \equiv \mathscr{C}_{\partial/\partial x^\mu|_x}$, we have $\eta_{\mu\nu} = \eta(\vec{e}_\mu, \vec{e}_\nu)$.

To vector $\vec{e}_\mu$ corresponds a form $\gamma_\mu = \eta_{\mu\nu}\gamma^\nu$. There exists an isomorphism between $\mathcal{M}$ and $\Lambda^1(\mathcal{M})$ given by $\mathcal{M} \ni \vec{a} \leftrightarrow a \in \Lambda^1(\mathcal{M})$, such that if $\vec{a} = a^\mu \vec{e}_\mu$, then $a = \eta_{\mu\nu}a^\mu\gamma^\nu$ and if $a = a_\mu\gamma^\mu$, then $\vec{a} = \eta^{\mu\nu}a_\mu\vec{e}_\nu$.

Definition 1.4. A scalar product of forms can be defined by

$$\Lambda^1(\mathcal{M}) \times \Lambda^1(\mathcal{M}) \ni (a, b) \mapsto a \cdot b \in R$$

such that if $\vec{a} \leftrightarrow a$ and $\vec{b} \leftrightarrow b$, then $a \cdot b = \eta(\vec{a}, \vec{b})$.

$(M, \mathcal{M}^*)$ equipped with the scalar product (1.1) is a representation of Minkowski spacetime. We denote by $Cl(\mathcal{M}) \approx Cl(1, 3) \approx \mathbf{H}(2)$ the spacetime algebra, i.e., the Clifford algebra (Lounesto, 1997) of $\mathcal{M}^*$ equipped with the scalar product defined by (1.1). As $\mathbb{R}$-space, $\mathcal{C}\ell(M) = \sum_{p=0}^{4} \Lambda^p(\mathcal{M})$.

The reciprocal basis of $\langle\gamma^\mu\rangle$ is $\langle\gamma_\mu\rangle$, $\gamma_\mu \cdot \gamma_\nu = \eta_{\mu\nu} = \gamma^\mu \cdot \gamma^\nu = \eta^{\mu\nu}$, $\gamma^\mu \cdot \gamma_\nu = \delta^\mu_\nu$, $\eta = \eta_{\mu\nu}\gamma^\mu \otimes \gamma^\nu = \eta^{\mu\nu}\gamma_\mu \otimes \gamma_\nu$.

The oriented affine space $(M, \mathcal{M}^*)$ (oriented by $\gamma^5 = \gamma^0 \wedge \gamma^1 \wedge \gamma^2 \wedge \gamma^3$) is a representation of the Minkowski manifold.

Definition 1.5 Let $\langle x^\mu\rangle$ be a global affine coordinate system for $(M, \mathcal{M}^*)$ relative to an arbitrary point $o \in M$. A position form associated to $x \in M$ is the form over $\mathcal{M}$, designed by the same letter, given by the following correspondence

$$M \ni x \leftrightarrows x = x^\mu\gamma_\mu \in \Lambda^1(\mathcal{M})$$

Definition 1.6. A smooth multiform field A on Minkowski spacetime is a multiform valued function of position form, $\Lambda^1(\mathcal{M}) \ni x \mapsto A(x) \in \Lambda(\mathcal{M})$.

Definition 1.7. Let $0 \leq p, q \leq 4$. A (p, q)-extensor t is a linear mapping

$$t\colon \quad \Lambda^p(\mathcal{M}) \to \Lambda^q(\mathcal{M})$$

The set of all (p, q)-extensors is denoted by $\text{ext}(\Lambda^p(\mathcal{M}), \Lambda^q(\mathcal{M}))$. A smooth (p, q)-extensor field t on Minkowski spacetime is a differentiable (p, q)-extensor-valued function of position form, $\Lambda^1(\mathcal{M}) \ni x \mapsto t_x \in ext(\Lambda^p(\mathcal{M}), \Lambda^q(\mathcal{M}))$.

Definition 1.8. Let a be a form. The a-directional derivative of a smooth multiform field X, denoted as $a \cdot \partial X$, is defined by

$$a \cdot \partial X = \lim_{\lambda\to 0} \frac{X(x + \lambda a) - X(x)}{\lambda} = \frac{d}{d\lambda} X(x + \lambda a)\Big|_{\lambda=0}$$

The γ_μ-directional derivative $\gamma_\mu \cdot \partial X$ coincides with the coordinate derivative $\partial X/\partial x^\mu$, $\partial_\mu \equiv \gamma_\mu \cdot \partial$.

Definition 1.9. The gradient, divergence, and curl of a smooth multiform field X are defined by

$$\text{gradient:} \qquad \partial X = \gamma^\mu(\partial_\mu X)$$

$$\text{divergence:} \qquad \partial \lrcorner X = \gamma^\mu \lrcorner (\partial_\mu X)$$

$$\text{curl:} \qquad \partial \wedge X = \gamma^\mu \wedge (\partial_\mu X)$$

$$\partial X = \partial \lrcorner X + \partial \wedge X$$

2. LAGRANGIAN MAPPINGS $(X, \partial * X) \mapsto \mathcal{L}(X, \partial * X)$

Let X be a smooth multiform field over the Minkowski spacetime M and let $*$ mean any one of the multiform products $(\lrcorner)$, $(\wedge)$, or Clifford product. Hestenes and Sobczyk (1984) introduced the multivector derivatives, the ordinary multiform derivatives $\partial * X$, i.e., the divergence, the curl, or the gradient of X. The dot product of Definition 1.5 is extended in a natural way to all $Cl(\mathcal{M})$ as follows: $\forall X, Y \in \Lambda(\mathcal{M})$, $X \cdot Y = \langle X\tilde{Y}\rangle_0$. Note that it is an operation different from the left and right contractions (Rodrigues *et al.*, 1995; Lounesto, 1999; Moya *et al.*, 2000).

Definition 2.1 (Lagrangian mapping). A differentiable scalar-valued function of two multiform variables $\mathcal{L}$: Im $X \times$ Im $\partial * X \to R$, where Im X means image of the multiform field X, etc., will be called the Lagrangian mapping (LM) associated to X.

Definition 2.2 Let X be any smooth multiform field; then $\hat{\mathcal{L}}[X]$ is a smooth scalar field defined by $\Lambda^1(\mathcal{M}) \ni x \mapsto \hat{\mathcal{L}}[X](x) \in R$ such that $\hat{\mathcal{L}}[X](x) = \mathcal{L}[X(x), \partial * X(x)]$.

The operator $\hat{\mathcal{L}}$ will be called the Lagrangian operator and the smooth scalar field $\hat{\mathcal{L}}[X]$ will be called the Lagrangian scalar field associated to X. In abuse of notation, in what follows, the Lagrangian mapping $\mathcal{L}$ and the Lagrangian scalar field $\hat{\mathcal{L}}[X]$ will be symbolized simply by $(X, \partial * X) \mapsto \mathcal{L}(X, \partial * X)$.

Definition 2.3. To any LM $(X\ \partial * X) \mapsto \mathcal{L}(X, \partial * X)$, the *action* for the multiform field X (on $U \subseteq M$) is the scalar, i.e., a real number,

$$S = \int_U \hat{\mathcal{L}}[X](x)\, d^4x = \int_U \mathcal{L}(X, \partial * X)\, d^4x$$

Take an arbitrary smooth multiform field A with the property $A = \langle A\rangle_X$ (i.e., A and X contain the same grades) such that it vanishes on the boundary ∂U (i.e., $A|_{\partial U} = O$) and take an open set $S_0 \subset R$ containing zero.

Definition 2.4. The following λ-parametrized smooth scalar field will be called the varied Langrangian:

$$\Lambda^1(\mathcal{M}) \times S_0 \ni (x, \lambda) \mapsto \hat{\mathcal{L}}[X + \lambda A](x) \in R \tag{2.1}$$

$$\hat{\mathcal{L}}[X + \lambda A](x) = \mathcal{L}[X(x) + \lambda A(x), \partial * X(x) + \lambda\partial * A(x)] \tag{2.2}$$

In abuse of notation, the varied Lagrangian sometimes will be denoted by $\mathcal{L}(X + \lambda A, \partial * X + \lambda\partial * A)$.

Definition 2.5 (Varied action). The following λ-parametrized scalar, i.e., an ordinary scalar function of the real variable λ, is called the varied action:

$$S_0 \ni \lambda \mapsto S(\lambda) \in R, \qquad S(\lambda) = \int_U \hat{\mathcal{L}}[X + \lambda A](x)\, d^4x$$

$$S(\lambda) = \int_U \mathcal{L}(X + \lambda A, \partial * X + \lambda \partial * A)\, d^4x$$

Definition 2.6. Given any smooth multiform field X and Lagrangian scalar field $\hat{\mathcal{L}}[X]$, the variational operator δ_X is defined by $\Lambda^1(\mathcal{M}) \ni x \mapsto \delta_X \hat{\mathcal{L}}[X](x) \in R$,

$$\delta_X \hat{\mathcal{L}}[X](x) = \frac{d}{d\lambda} \hat{\mathcal{L}}[X + \lambda A](x)\big|_{\lambda=0}$$

$$= \frac{d}{d\lambda} \mathcal{L}[X(x) + \lambda A(x), \partial * X(x) + \lambda \partial * A(x)]\big|_{\lambda=0}$$

$$\delta_X \mathcal{L}(X, \partial * X) = \frac{d}{d\lambda} \mathcal{L}(X + \lambda A, \partial * X + \lambda \partial * A)\big|_{\lambda=0}$$

In Lagrangian field theory, the dynamics of a multiform field X is supposed to satisfy the condition of stationary action *AP*,

$$S'(0) = 0, \qquad \forall A \quad \text{such that} \quad A\big|_{\partial U} = O$$

$$\int_U \delta_X \mathcal{L}(X, \partial * X)\, d^4x = 0, \forall A \quad \text{such that} \quad A\big|_{\partial U} = O$$

The *AP* implies the Euler-Lagrange equation (ELE) for the multiform field X, i.e., the field equation for X.

Proposition 2.7. Given a dynamical variable X on $U \subseteq M$ and a LM $(X, \partial * X) \mapsto \mathcal{L}(X, \partial * X)$, where $*$ is respectively (a) $\lrcorner$, (b) $\wedge$, or (c) the Clifford product, the *AP* implies for the cases (a)–(c), respectively, the following ELEs:

(a) $\partial_X \mathcal{L}(X, \partial \lrcorner X) - \partial \wedge \partial_{\partial \lrcorner X} \mathcal{L}(X, \partial \lrcorner X) = O$
(b) $\partial_X \mathcal{L}(X, \partial \wedge X) - \partial \lrcorner \partial_{\partial \wedge X} \mathcal{L}(X, \partial \wedge X) = O$
(c) $\partial_X \mathcal{L}(X, \partial X) - \partial \partial_{\partial X} \mathcal{L}(X, \partial X) = O$

Proof. We prove case (c). The X-variation of $\mathcal{L}(X, \partial X)$ yields

$$\delta_X \mathcal{L}(X, \partial X) = A \cdot \partial_X \mathcal{L}(X, \partial X) + \partial A \cdot \partial_{\partial X} \mathcal{L}(X, \partial X)$$

Using the identity (A.3), we have

$$\delta_X \mathcal{L}(X, \partial X) = A \cdot [\partial_X \mathcal{L}(X, \partial X) - \partial \partial_{\partial X} \mathcal{L}(X, \partial X)]$$
$$+ \partial \cdot [\partial_a (aA) \cdot \partial_{\partial X} \mathcal{L}(X, \partial X)]$$

The *AP* yields

$$\int_U A \cdot (\partial_X \mathcal{L} - \partial\partial_{\partial X}\mathcal{L})\, d^4x + \int_U \partial \cdot [\partial_a|(aA) \cdot \partial_{\partial X}\mathcal{L}]\, d^4x = 0 \quad (2.3)$$

for all A such that $A|_{\partial U} = O$.

Using the Gauss–Stokes theorem with the boundary condition $A|_{\partial U} = O$, the second term gives

$$\int_U \partial \cdot [\partial_a(aA) \cdot \partial_{\partial X}\mathcal{L}]\, d^4x = \oint_{\partial U} \gamma^\mu \cdot [\partial_a(aA) \cdot \partial_{\partial X}\mathcal{L}]\, d^3S_\mu$$

$$= \oint_{\partial U} A \cdot (\gamma^\mu \partial_{\partial X}\mathcal{L})\, d^3S_\mu = 0 \qquad (2.4)$$

Putting (2.4) into (2.3) gives

$$\int_U A \cdot [\partial_X \mathcal{L}(X, \partial X) - \partial\partial_{\partial X}\mathcal{L}(X, \partial X)]\, d^4x = 0, \quad \forall A$$

and due to the arbitrariness of A, we get

$$\partial_X \mathcal{L}(X, \partial X) - \partial\partial_{\partial X}\mathcal{L}(X, \partial X) = O \quad \blacksquare$$

3. LAGRANGIAN MAPPINGS $(X, \mathcal{D} * X) \mapsto \mathcal{L}(X, \mathcal{D} * X)$

Let X be a smooth multiform field on $(U \subseteq M, \mathcal{M}^*)$ and let h be an invertible (1,1)-extensor field (h_x: $\mathcal{M}^* \ni x \mapsto ext(\Lambda^1(\mathcal{M}), \Lambda^1(\mathcal{M}))$) called the *gauge metric extensor field* (which is a representation of the gravitational field in the most general possible gravitational theory over Minkowski spacetime). Also, define $h^* = (h^{-1})^\dagger = (h^\dagger)^{-1}$. Consider the operators $\mathcal{D}*$, where $*$ means any multiform product ($\lrcorner$), ($\wedge$), or the Clifford product acting on the set of smooth multivector fields. They are called the *h-divergence* $\mathcal{D} \lrcorner X \equiv h^\star(\partial_a) \lrcorner \mathcal{D}_a X$, the *h-curl* $\mathcal{D} \wedge X \equiv h^\star(\partial_a) \wedge \mathcal{D}_a X$, and the *h-gradient* $\mathcal{D}X \equiv h^\star(\partial_a)\mathcal{D}_a X$.

$\mathcal{D}_a X$ is a directional covariant derivative obtained from the Levi-Civita directional covariant derivative D_a, $\mathcal{D}_a X = h(D_a h^{-1}(X))$ studied in the theory of connections in Fernández *et al.*, 2000b; Moya *et al.*, 2000a, b),

$$\mathcal{D}_a X = a \cdot \partial X + \Omega(a) \times X$$

Ω is called the second connection extensor field, Ω_x: $\Lambda^1(\mathcal{M}) \to \Lambda^2(\mathcal{M})$, $\forall x \in \mathcal{M}^*$, $\Omega_x(a) = -\frac{1}{2}\partial_{n(x)} \wedge \mathcal{D}_a n(x)$.

In theories which make use of the gauge-covariant derivative concept, the action for the multiform field X [on $(U \subseteq M, \mathcal{M}^*)$], with dynamics given by a LM $(X, \mathcal{D} * X) \mapsto \mathcal{L}(X, \mathcal{D} * X)$, the action is postulated to be the scalar

$$S = \int_U \mathcal{L}(X, \mathcal{D} * X)\, d^4x$$

Take an arbitrary smooth multiform field A with the property $A = \langle A \rangle_X$ such that $A|_{\partial U} = O$ and take an open set $S_0 \subset R$ containing zero.

Definition 3.1. The A-varied action for the multivector field X (on $U \subseteq M$) is the λ-parametrized scalar

$$S(\lambda) = \int_U \mathcal{L}(X + \lambda A, \mathcal{D} * X + \lambda \mathcal{D} * A)\, d^4x$$

The dynamics of the multiform field X is supposed to satisfy the *AP*

$$S'(0) = 0, \qquad \forall A \quad \text{such that} \quad A|_{\partial U} = O$$

$$\int_U \delta_x \mathcal{L}(X, \mathcal{D} * X)\, d^4x = 0, \qquad \forall A \quad \text{such that} \quad A|_{\partial U} = O$$

where

$$\delta_x \mathcal{L}(X, \mathcal{D} * X) = \frac{d}{d\lambda} \mathcal{L}(X + \lambda A, \mathcal{D} * X + \lambda \mathcal{D} * A)|_{\lambda=0}$$

is the X-variation of $\mathcal{L}(X, \mathcal{D} * X)$.

Proposition 3.2. Given a dynamical variable X and an *LM*

$$(X, \mathcal{D} * X) \mapsto \mathcal{L}(X, \mathcal{D} * X) = \det(h) l(X, \mathcal{D} * X)$$

where $*$ is respectively (a) $\lrcorner$, (b) $\wedge$, or (c) the Clifford product, the AP implies for the cases (a)–(c) the following ELEs:

(a) $\partial_X l(X, \mathcal{D} \lrcorner, X) - \mathcal{D} \wedge \partial_{\mathcal{D} \lrcorner X}\, l(X, \mathcal{D} \lrcorner X) = O$

(b) $\partial_X l(X, \mathcal{D} \wedge X) - \mathcal{D} \lrcorner\, \partial_{\mathcal{D} \wedge X}\, l(X, \mathcal{D} \wedge X) = O$

(c) $\partial_X l(X, \mathcal{D} X) - \mathcal{D} \partial_{\mathcal{D} X}\, l(X, \mathcal{D} X) = O$

Proof. We prove case (b). Using the multiform identity (A.7), the X-variation of $\mathcal{L}(X, \mathcal{D} \wedge X)$ yields

$$\begin{aligned} \delta_X \mathcal{L}(X, \mathcal{D} \wedge X) &= \det(h)[A \cdot \partial_X l(X, \mathcal{D} \wedge X) + \mathcal{D} \wedge A \cdot \partial_{\mathcal{D} \wedge X} l(X, \mathcal{D} \wedge X)] \\ &= \det(h) A \cdot [\partial_X l(X, \mathcal{D} \wedge X) - \mathcal{D} \lrcorner\, \partial_{\mathcal{D} \wedge X} l(X, \mathcal{D} \wedge X)] \\ &= \partial \cdot [\det(h) \partial_a (h^\star(\text{a}) \wedge A) \cdot \partial_{\mathcal{D} \wedge X} l(X, \mathcal{D} \wedge X)] \end{aligned}$$

The *AP* action yields

$$\int_U \det(h)A \cdot (\partial_x l - \mathscr{D} \lrcorner\, \partial_{\mathscr{D}\wedge X} l(\, d^4x + \int_U \partial \cdot [\det(h)\partial_a(h^\star(a) \wedge A) \cdot \partial_{\mathscr{D}\wedge X} l] \, d^4x = 0 \tag{3.1}$$

for all A such that $A|_{\partial U} = O$.

Using the Gauss–Stokes theorem with the boundary condition $A|_{\partial U} = O$, the second term gives

$$\begin{aligned}\int_U \partial \cdot [\det(h)\partial_a(h^\star(a) \wedge A) \cdot \partial_{\mathscr{D}\wedge X} l] \, d^4x &= \oint_{\partial U} \det(h)\gamma^\mu \cdot [\partial_a(h^\star(a) \wedge A) \cdot \partial_{\mathscr{D}\wedge X} l] \, d^3S_\mu \\ &= \oint_{\partial U} \det(h)A \cdot [h^\star(\gamma^\mu) \lrcorner\, \partial_{\mathscr{D}\wedge X} l] \, d^3S_\mu = 0\end{aligned} \tag{3.2}$$

Putting Eq. (3.2) into Eq. (3.1), we have

$$\int_U \det(h)A \cdot [\partial_X l(X, \mathscr{D} \wedge X) - \mathscr{D} \lrcorner\, \partial_{\mathscr{D}\wedge X} l(X, \mathscr{D} \wedge X)] \, d^4x = 0 \qquad \text{for all} \quad A$$

and due to the arbitrariness of A, we finally get

$$\partial_X l(X, \mathscr{D} \wedge X) - \mathscr{D} \lrcorner\, \partial_{\mathscr{D}\wedge X} l(X, \mathscr{D} \wedge X) = O$$

The proofs of (a) and (c) can be obtained by using the multiform identities (A.6) and (A.8). ■

4. LAGRANGIAN MAPPING $(\psi, \mathscr{D}^s\psi) \mapsto \mathscr{L}(\psi, \mathscr{D}^s\psi)$

Let ψ be a smooth Dirac–Hestenes spinor field (DHSF) on $(U \subseteq M, \mathcal{M}^*)$. We can take the *gauge spinor derivative* (Rodrigues *et al.*, 1995; Fernández *et al.*, 2000b) $\mathscr{D}^s\psi \equiv h^\star(\partial_a)\mathscr{D}^s_a\psi$ [recall that $\mathscr{D}^s_a\psi \equiv a \cdot \partial\psi + \frac{1}{2}\Omega(a)\psi$ is the *directional spinor derivative*] and consider an LM$(\psi, \mathscr{D}^s\psi) \mapsto \mathscr{L}(\psi, \mathscr{D}^s\psi)$.

Definition 4.1. The action for a DHSF ψ (on $U \subseteq M$) is

$$S = \int_U \mathscr{L}(\psi, \mathscr{D}^s\psi) \, d^4x.$$

If we take an arbitrary smooth DHSF η such that $\eta|_{\partial U} = O$ then the so-called ψ-variation of $\mathscr{L}(\psi, \mathscr{D}^s\psi)$ is

$$\delta_\psi \mathcal{L}(\psi, \mathcal{D}^s\psi) = \frac{d}{d\lambda} \mathcal{L}(\psi + \lambda\eta, \mathcal{D}^s\psi + \lambda\mathcal{D}^s\eta)\big|_{\lambda=0}$$

Proposition 4.2. Given a DHSF ψ as dynamical variable and a LM $(\psi, \mathcal{D}^s\psi) \mapsto \mathcal{L}(\psi, \mathcal{D}^s\psi)$, the *AP*

$$\int_U \delta\psi\mathcal{L}(\psi, \mathcal{D}^s\psi)\, d^4x = 0$$

for all η such that $\eta|_{\partial U} = O$ implies the ELE

$$\partial_\psi l(\psi, \mathcal{D}^s\psi) - \mathcal{D}^s\partial_{\mathcal{D}^s\psi} l(\psi, \mathcal{D}^s\psi) = O$$

Proof. The ψ-variation of $\mathcal{L}(\psi\ \mathcal{D}^s\psi)$ is

$$\delta\psi\mathcal{L}(\psi, \mathcal{D}^s\psi) = \det(h)\eta \cdot [\partial_\psi l(\psi, \mathcal{D}^s\psi) + \mathcal{D}^s\eta \cdot \partial_{\mathcal{D}^s\psi} l(\psi, \mathcal{D}^s\psi)]$$

and using the multiform identity (A.12), we have

$$\begin{aligned}\delta\psi\mathcal{L}(\psi, \mathcal{D}^s\psi) = {} & \det(h)[\eta \cdot \partial_\psi l(\psi, \mathcal{D}^s\psi) - \mathcal{D}^s\partial_{\mathcal{D}^s\psi} l(\psi, \mathcal{D}^s\psi)] \\ & + \partial \cdot [\det(h)\partial_a(h^\star(a)\eta) \cdot \partial_{\mathcal{D}^s\psi} l(\psi, \mathcal{D}^s\psi)]\end{aligned}$$

The *AP* can be written for all η such that $\eta|_{\partial U} = O$

$$\begin{aligned}&\int_U \det(h)\eta \cdot (\partial_\psi l - \mathcal{D}^s\partial_{\mathcal{D}^s\psi} l)\, d^4x \\ &\quad + \int_U \partial \cdot [\det(h)\partial_a(h^\star(a)\eta) \cdot \partial_{\mathcal{D}^s\psi} l]\, d^4x = 0 \end{aligned} \tag{4.1}$$

The second term can be integrated using the Gauss–Stokes theorem with the boundary condition $\eta|_{\partial U} = O$

$$\begin{aligned}&\int_U \partial \cdot [\det(h)\partial_a(h^\star(a)\eta) \cdot \partial_{\mathcal{D}^s\psi} l]\, d^4x \\ &\quad = \oint_{\partial U} \gamma^\mu \cdot [\det(h)\partial_a(h^\star(a)\eta) \cdot \partial_{\mathcal{D}^s\psi} l]\, d^3S_\mu \\ &\quad = \oint_{\partial U} \det(h)\eta \cdot [h^\star(\gamma^\mu)\partial_{\mathcal{D}^s\psi} l]\, d^3S_\mu = 0\end{aligned} \tag{4.2}$$

Putting (4.2) into (4.1), we get

$$\int_U \det(h)\eta \cdot [\partial_\psi l(\psi, \mathcal{D}^s\psi) - \mathcal{D}^s\partial_{\mathcal{D}^s\psi} l(\psi, \mathcal{D}^s\psi)]\, d^4x = 0 \qquad \text{for all} \quad \eta$$

Since η is arbitrary, it follows that

$$\partial_\psi l(\psi, \mathscr{D}^s\psi) - \mathscr{D}^s \partial_{\mathscr{D}^s\psi} l(\psi, \mathscr{D}^s\psi) = O \quad \blacksquare$$

5. EXAMPLES

5.1. Maxwell and Dirac–Hestenes Lagrangians

(a) The Lagrangian associated to the Maxwell field A: $\mathscr{M}^* \rightarrow \Lambda^1(\mathscr{M})$, i.e., the electromagnetic potential generated by an electric charge current density J: $\mathscr{M}^* \rightarrow \Lambda^1(\mathscr{M})$, is

$$\mathscr{L}(A, \partial \wedge \mathrm{A}) = -\frac{1}{2\mu_0}(\partial \wedge A) \cdot (\partial \wedge A) - A \cdot J$$

The Euler–Lagrange equation is

$$\partial_A \mathscr{L}(A, \partial \wedge A) - \partial \lrcorner\, \partial_{\partial\wedge A} \mathscr{L}(A, \partial \wedge A) = O$$

Then, the Maxwell field A and the Faraday field $F = \partial \wedge A$ satisfy the equations

$$\partial \lrcorner\, (\partial \wedge A) = \mu_0 J, \qquad \partial F = \mu_0 J \tag{5.1}$$

The second equation in (5.1) is the Maxwell equation in the spacetime calculus formalism (Hestenes, 1966).

(b) DHSF are certain equivalence classes of even sections of $Cl(\mathscr{M})$. For details, see Rodrigues *et al.* (1995). In quantum mechanics, the Lagrangian associated to the DHSF ψ: $\mathscr{M}^* \rightarrow \Lambda^0(\mathscr{M}) + \Lambda^2(\mathscr{M}) + \Lambda^4(\mathscr{M})$, corresponding to a particle with mass m, electric charge e, and spin $\frac{1}{2}$ (i.e., a Dirac particle) in interaction with the Maxwell field A, is

$$\mathscr{L}(\psi, \partial\psi) = \hbar(\partial\psi \mathbf{i}\gamma_3) \cdot \psi - e(A\psi\gamma_0) \cdot \psi - mc\psi \cdot \psi, \qquad \mathbf{i} = \gamma_0\gamma_1\gamma_2\gamma_3$$

A thoughtful study of the Dirac–Hestenes Lagrangian which shows hidden assumptions in the usual presentation can be found in De Leo *et al.* (1999). To get the ELE, we need

$$\partial_\psi \mathscr{L}(\psi, \partial\psi) = \hbar\partial\psi\mathbf{i}\gamma_3 - 2eA\psi\gamma_0 - 2mc\psi$$

$$\partial_{\partial\psi} \mathscr{L}(\psi, \partial\psi) = -\hbar\partial_{\partial\psi}(\partial\psi \cdot \psi\mathbf{i}\gamma_3) = -\hbar\psi\mathbf{i}\gamma_3$$

where the following multiform derivative formulas have been used:

$$\partial_X(X \cdot X) = 2X, \qquad \partial_X(X \cdot Y) = \langle Y \rangle_X,$$

$$\partial_X[(YXZ) \cdot X] = \langle YXZ + \tilde{Y}X\tilde{Z} \rangle_X$$

The DHSF ψ satisfies the Dirac equation, called the Dirac–Hestenes equation (Hestenes, 1996) in the spacetime calculus formalism,

$$\hbar \partial \psi \mathbf{i} \sigma_3 - eA\psi = mc\psi\gamma_0, \qquad \sigma_3 = \gamma_3\gamma_0$$

5.2. Lagrangian for Maxwell and Dirac–Hestenes Fields on a Gravitational Field Background

In the flat-spacetime formulation of the most general possible gravitational field theory (which includes curvature and torsion), this field is described by an invertible (1,1)–extensor field h, Section 3.

(a) The *dynamics* of a Maxwell field A generated by an electric charge current density J moving in a background gravitational field is postulated to derive from the *AP* and the following Lagrangian:

$$\mathcal{L}(A, \mathcal{D} \wedge A) = \det(h)\left[-\frac{1}{2\mu_0} (\mathcal{D} \wedge A) \cdot (\mathcal{D} \wedge A) - A \cdot J \right]$$

Using the identities in the Appendix, we obtain that A and $F = \mathcal{D} \wedge A$ satisfy

$$\mathcal{D} \lrcorner (\mathcal{D} \wedge A) = \mu_0 J, \qquad \mathcal{D}F = \mu_0 J$$

(b) The dynamics of a DHSF ψ corresponding to a particle with mass m, electric charge e, and spin $\frac{1}{2}$ (i.e., a Dirac particle) is supposed to be governed by the *AP* with Lagrangian

$$\mathcal{L}(\psi, \mathcal{D}^s\psi) = \det(h)[\hbar(\mathcal{D}^s\psi\mathbf{i}\gamma_3) \cdot \psi - e(A\psi\gamma_0) \cdot \psi - mc\psi \cdot \psi]$$

Using the identities of Appendix, we get that the DHSF ψ satisfies

$$\hbar \mathcal{D}^s \psi \mathbf{i} \sigma_3 - eA\psi = mc\psi\gamma_0$$

APPENDIX. FUNDAMENTAL IDENTITIES IN LAGRANGIAN FORMALISM

Proposition A.1. For all smooth multiform fields X and Y and 1-form field a, we have

$$(\partial \lrcorner X) \cdot Y + X \cdot (\partial \wedge Y) = \partial \cdot [\partial_a(a \lrcorner X) \cdot Y] \tag{A.1}$$

$$(\partial \wedge X) \cdot Y + X \cdot (\partial \lrcorner Y) = \partial \cdot [\partial_a(a \wedge X) \cdot Y] \tag{A.2}$$

$$(\partial X) \cdot Y + X \cdot (\partial Y) = \partial \cdot [\partial_a(aX) \cdot Y] \tag{A.3}$$

These identities are necessary in the derivation from the *AP* of the ELE equations for a multiform field X with dynamics governed by the LM $(X, \partial \lrcorner X) \mapsto \mathcal{L}(X, \partial \lrcorner X)$, $(X, \partial \wedge X) \mapsto \mathcal{L}(X, \partial \wedge X)$, or $(X, \partial X) \mapsto \mathcal{L}(X, \partial X)$.

Proof. In order to prove the identity (A.1), we use the definitions of divergence and curl of a multiform field and the identity $(a \lrcorner B) \cdot C = B \cdot (a \wedge C)$, where a is a 1-form and B, C are multiforms,

$$(\partial \lrcorner X) \cdot Y + X \cdot (\partial \wedge Y) = (\gamma^\mu \lrcorner \gamma_\mu \cdot \partial X) \cdot Y + X \cdot (\gamma^\mu \wedge \gamma_\mu \cdot \partial Y)$$
$$= \gamma_\mu \cdot \partial(\gamma^\mu \lrcorner X) \cdot Y + (\gamma^\mu \lrcorner X) \cdot (\gamma_\mu \cdot \partial Y)$$
$$= \gamma_\mu \cdot \partial(\gamma^\mu \lrcorner X) \cdot Y \qquad \text{(A.4)}$$

but it is not difficult to transform the right side of (A.4) into a divergence of a 1-form field,

$$\gamma_\mu \cdot \partial(\gamma^\mu \lrcorner X) \cdot Y = \gamma_\beta \cdot \partial[\gamma^\beta \cdot \gamma_\mu(\gamma^\mu \lrcorner X) \cdot Y]$$
$$= \gamma^\beta \cdot \gamma_\beta \cdot \partial[\gamma_\mu(\gamma^\mu \lrcorner X) \cdot Y]$$
$$= \partial \cdot [\partial_a(a \lrcorner X) \cdot Y] \qquad \text{(A.5)}$$

Putting (A.4) into (A.5) complete the proof.

The identity (A.2) can be proved by using (A.1) and the identity $(a \lrcorner B) \cdot C = B\cdot (a \wedge C)$,

$$(\partial \wedge X) \cdot Y + X \cdot (\partial \lrcorner Y) = (\partial \neg Y) \cdot X + Y \cdot (\partial \wedge X) = \partial \cdot [\partial_a(a \lrcorner Y) \cdot X]$$
$$= \partial \cdot [\partial_a Y \cdot (a \wedge X)] = \partial \cdot [\partial_a(a \wedge X) \cdot Y]$$

Identity (A.3) can be proved by adding (A.1) and (A.2). ■

Proposition A.2. For all smooth multiform fields X and Y and 1-form field a, we have

$$(\mathcal{D} \lrcorner X) \cdot Y + X \cdot (\mathcal{D} \wedge Y) = \det(h^{-1})\partial \cdot [\det(h)\partial_a(h^\star(a) \lrcorner X) \cdot Y] \qquad \text{(A.6)}$$
$$(\mathcal{D} \wedge X) \cdot Y + X \cdot (\mathcal{D} \lrcorner Y) = \det(h^{-1})\partial \cdot [\det(h)\partial_a(h^\star(a) \wedge X) \cdot Y] \qquad \text{(A.7)}$$
$$(\mathcal{D}X) \cdot Y + X \cdot (\mathcal{D}Y) = \det(h^{-1})\partial \cdot [\det(h)\partial_a(h^\star(a)X) \cdot Y] \qquad \text{(A.8)}$$

These multiform identities are necessary in order to derive from the *AP* the ELEs for multiform fields X with Lagrangian mappings $(X, \mathcal{D} \lrcorner X) \mapsto \mathcal{L}(X, \mathcal{D} \lrcorner X)$, or $(X, \mathcal{D} \wedge X) \mapsto \mathcal{L}(X, \mathcal{D} \wedge X)$, or $(X, \mathcal{D}\, X) \mapsto \mathcal{L}\,(X, \mathcal{D}\, X)$.

Proof. To prove the identity (A.6), we shall need to use two multiform identities relating the gauge-covariant divergence and the ordinary divergence, and the gauge-covariant curl and the ordinary curl, respectively (Fernández *et al.*,2000b),

$$\mathcal{D} \lrcorner A = \det(h^{-1})\underline{h}[\partial \lrcorner \det(h)\underline{h}^{-1}(A)], \qquad \mathcal{D} \wedge A = \underline{h}^\star[\partial \wedge \underline{h}^\dagger(A)]$$

and the multiform identity (A.1) above. We have

$$
\begin{aligned}
(\mathcal{D} \lrcorner X) \cdot Y + X \cdot (\mathcal{D} \wedge Y) &= \det(h^{-1})\underline{h}[\partial \lrcorner \det(h)\underline{h}^{-1}(X)] \cdot Y \\
&\quad + X \cdot \underline{h}^{\star}[\partial \wedge h^{\dagger}(Y)] \\
&= \det(h^{-1})\{[\partial \lrcorner \det(h)\underline{h}^{-1}(X)] \cdot \underline{h}^{\dagger}(Y) \\
&\quad + \det(h)\underline{h}^{-1}(X) \cdot [\partial \wedge \underline{h}^{\dagger}(Y)]\} \\
&= \det(h^{-1})\partial \cdot [\partial_a(a \lrcorner \det(h)\underline{h}^{-1}(X)) \cdot \underline{h}^{\dagger}(Y)] \\
&= \det(h^{-1})\partial \cdot [\det(h)\partial_a(a \lrcorner \underline{h}^{-1}(X)) \cdot \underline{h}^{\dagger}(Y)]
\end{aligned} \tag{A.9}
$$

Given a (1,1)-extensor t over $\Lambda(\mathcal{M})$,its extension $\underline{t}$, a general extensor over $\Lambda(\mathcal{M})$, is defined by

$$
\underline{t}(X) = 1 \cdot X + \sum_{k=1}^{n} \frac{1}{k!} t(\gamma^{\mu_1}) \wedge \ldots \wedge t(\gamma^{\mu_k})(\gamma_{\mu_1} \wedge \ldots \wedge \gamma_{\mu_k}) \cdot X
$$

Using the algebraic identity $a \lrcorner \underline{t}(B) = \underline{t}[\underline{t}^{\dagger}(a) \lrcorner B]$, where $\underline{t}$ is the extension of a (1,1)-extensor t, a is a 1-form, and B is a multiform, we can write

$$
\begin{aligned}
[a \lrcorner \underline{h}^{-1}(X)] \cdot \underline{h}^{\dagger}(Y) &= \underline{h}^{-1}[\underline{h}^{\star}(a) \lrcorner X] \cdot \underline{h}^{\dagger}(Y) \\
&= \underline{hh}^{-1}[\underline{h}^{\star}(a) \lrcorner X] \cdot Y = [\underline{h}^{\star}(a) \lrcorner X] \cdot Y
\end{aligned} \tag{A.10}
$$

Putting (A.10) into the right side of (A.9) completes the proof.

The second identity (A.7) can be proved using (A.6), the algebraic identity $(a \lrcorner B) \cdot C = B \cdot (a \wedge C)$, and following analogous steps just used in demonstrating the identity(A.2). The third identity (A.8) can be proved easily by adding (A.6) and (A.7). ■

Proposition A.3. For all smooth DHSF ψ and φ, we have

$$
(\mathcal{D}^s\psi) \cdot \varphi + \psi \cdot (\mathcal{D}^s\varphi) = (\mathcal{D}\psi) \cdot \varphi + \psi \cdot (\mathcal{D}\varphi) \tag{A.11}
$$

$$
(\mathcal{D}^s\psi) \cdot \varphi + \psi \cdot (\mathcal{D}^s\varphi) = \det(h^{-1})\partial \cdot [\det(h)\partial_a(h^{\star}(a)\psi) \cdot \varphi] \tag{A.12}
$$

Proof. To prove (A.11), note that for a smooth spinor field ψ, we have

$$
\mathcal{D}\psi = \mathcal{D}^s\psi - \tfrac{1}{2}h^{\star}(\partial_a)\psi\Omega(a)
$$

With the DHSF ψ and φ, we can write

$$\begin{aligned}(\mathcal{D}^s\psi)\cdot\varphi + \psi\cdot(\mathcal{D}^s\varphi) &= [\mathcal{D}\psi + \tfrac{1}{2}h^\star(\partial_a)\psi\Omega(a)]\cdot\varphi \\ &\quad + \psi\cdot[\mathcal{D}\varphi + \tfrac{1}{2}h^\star(\partial_a)\varphi\Omega(a)] \\ &= (\mathcal{D}\psi)\cdot\varphi + \psi\cdot(\mathcal{D}\varphi) \\ &\quad - \tfrac{1}{2}h^\star(\partial_a)\cdot\varphi\Omega(a)\tilde{\psi} - \tfrac{1}{2}h^\star(\partial_a)\cdot\psi\,\Omega(a)\tilde{\varphi}\end{aligned}$$

The last two terms yield zero,

$$\begin{aligned}\tfrac{1}{2}h^\star(\partial_a)\cdot[\varphi\Omega(a)\tilde{\psi} + \psi\Omega(a)\varphi] &= \tfrac{1}{2}h^\star(\partial_a)\cdot[\,\varphi\Omega(a)\tilde{\psi} - (\varphi\Omega(a)\tilde{\psi})^{\sim}] \\ &= \tfrac{1}{2}h^\star(\partial_a)\cdot 2\langle\varphi\Omega(a)\tilde{\psi}\rangle_2 = 0\end{aligned}$$

Equation (A.12) follows from (A.11) and (A.8). ■

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